GROUP CLASSIFICATION OF A FAMILY OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. We find the group of equivalence transformations for equations of the form $y'' = A(x)y' + F(y)$, where A and F are arbitrary functions. We then give a complete group classification of these families of equations, using a direct method of analysis, together with the equivalence transformations.

1. INTRODUCTION

The group classification problem of equations of the form

$$y'' = F(x,y,y') \tag{1.1}$$

was first considered by Lie [1], who showed that the symmetry group of all these equations is at most eight-dimensional and that this maximum is reached only if the equation can be mapped by a point transformation to a second-order linear ordinary differential equation (ODE). More recently, Ovsyannikov considered in [2] the problem of group classification of a much restricted form of the equation considered by Lie, namely the equation of the form

$$y'' = F(x,y). \tag{1.2}$$

This study showed, amongst others, that in the nonlinear case, which occurs if and only if $F = F(x,y)$ is not linear in y, the symmetry algebra has dimension at most two, except when F in (1.2) can be reduced by an equivalence transformation to $F = \pm y^{-3}$, in which case the nonlinear equation has a symmetry algebra of maximal dimension three.

Equations of the form (1.1) containing a term linear in y', and of the form

$$y'' = \frac{M}{x} y' + g(x)F(y), \tag{1.3}$$

where $g(x)$ and $F(y)$ are some given functions of x and y respectively, and M is a constant, also appear frequently in the mathematical physics literature. Eq. (1.3) is referred to as an Emden-Fowler type equation [3, 4], and it is reduced for $M = 2$, $g(x) = 1$, and $F(y) = y^n$ to the so-called *standard* Lane-Emden equation of index n, proposed by Lane [5] and studied in detail by Emden [6] and Fowler [7]. It has been used as a model for the dynamics of a spherical cloud of gas acting under mutual attraction of its molecules [3]. Equation (1.3) with $g(x) = 1$ is usually called the generalized Lane-Emden equation and for special cases of $F(y)$, it has also been used as a model for various phenomena in physics and astrophysics, such as the stellar structure, the thermionic currents, and the dynamics of isothermal spheres [8, 3, 4, 9].

Special cases as well as slightly modified forms of (1.3) have been considered for symmetry analysis and first integrals or exact solutions [10, 11, 12]. However, as far as the

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group classification of the Emden-Fowler type equation is concerned, only a classification of Noether point symmetries of the generalized Lane-Emden equation has been considered [13], but only for various (and not arbitrary) functions F. It should also be noted that the problem of determination of solutions of (1.3) by analytic approximations using the Adomian decomposition method, and incorporating a singularity analysis was considered in [8, 14].

The purpose of this paper is to provide a group classification of the equation

$$y'' = A(x)y' + F(y), \tag{1.4}$$

in which A and F are arbitrary functions of the independent variable x and the dependent variable y, respectively. This is a modified form of (1.3) in which the coefficient of y' is an arbitrary function and $g(x) = 1$. We find the group of equivalence transformations of this equation, that is, the largest group of point transformations that preserves the form of the equation. Next, we obtain the group classification of the equation, based on a direct analysis and using the equivalence transformations. It is shown in particular that in the nonlinear case, which occurs if and only if F is nonlinear, the maximal dimension of the symmetry algebra is three. Moreover, it is also clearly shown that any symmetry exists only for canonical forms of F of the form

$$f, \quad \mu e^y + f, \quad \mu y \ln(y) + f, \quad \mu \ln(y) + f, \quad \text{and } y^n + f, \quad n \neq 0, 1, \tag{1.5}$$

where $\mu \neq 0$ is a constant and f is a linear function of y.

2. Equivalence group

We shall say that an invertible point transformation of the form

$$x = S(z,w), \qquad y = T(z,w) \tag{2.1}$$

is an equivalence transformation of (1.4) if it transforms the latter equation into an equation of the same form, that is, into an equation of the form

$$w'' = B(z)w' + H(w), \tag{2.2}$$

where $B(z)$ and $H(w)$ are the new arbitrary functions, and where $w' = dw/dz$. In this case the two equations (1.4) and (2.2) are said to be equivalent. The equivalence group G of (1.4) is the largest Lie pseudo-group of transformations of the form (2.1) that preserves the form of the equation. By a result of Lie [15], the resulting transformations of the arbitrary functions A and F also form a Lie pseudo-group of transformations, which in the actual case can be put into the form

$$A = \chi(z,w,B,H), \qquad F = \zeta(z,w,B,H), \tag{2.3}$$

for certain functions χ and ζ which may be read-off from expressions of the transformed coefficients once the defining functions S and T of G are known. By writing down the transformation of (1.4) under (2.1), we obtain an equation in S and T which is rearranged by an expansion into powers of w'. Then, using the fact that S and T do not depend explicitly on the derivatives of w with respect to z, and assuming that (2.1) maps (1.4) to (2.2), the transformed equation is reduced to the following set of four equations in which $B(z)w' + H(w)$ is substituted for w'', and where $\delta = (S_z T_w - S_w T_z)$ is the Jacobian of the

change of variables (2.1) :

$$-FS_z^3+H\delta-T_z\left(AS_z^2+S_{z,z}\right)+S_zT_{z,z}=0 \tag{2.4a}$$

$$\begin{aligned}&-3FS_wS_z^2+BS_zT_w-AS_z^2T_w-BS_wT_z-2AS_wS_zT_z\\&-2T_zS_{z,w}-T_wS_{z,z}+2S_zT_{z,w}+S_wT_{z,z}=0\end{aligned} \tag{2.4b}$$

$$-3FS_w^2S_z-AS_w\left(2S_zT_w+S_wT_z\right)-T_zS_{w,w}-2T_wS_{z,w}+S_zT_{w,w}+2S_wT_{z,w}=0 \tag{2.4c}$$

$$-FS_w^3-T_w\left(AS_w^2+S_{w,w}\right)+S_wT_{w,w}=0. \tag{2.4d}$$

From (2.4d), it follows that $S_w^3=0$, on account of the arbitrariness of F, and so $S=S(z)$. When this last equality is substituted into (2.4), (2.4c) is reduced to $S_zT_{w,w}=0$, which shows that $T=\alpha(z)w+\beta(z)$, for some functions α and β. With these expressions for S and T, the first two equations of (2.4) are reduced to

$$H\alpha S_z-FS_z^3-AS_z^2\left(w\alpha_z+\beta_z\right)-\left(w\alpha_z+\beta_z\right)S_{z,z}+S_z\left(w\alpha_{z,z}+\beta_{z,z}\right)=0 \tag{2.5a}$$

$$B\alpha S_z-A\alpha S_z^2+2S_z\alpha_z-\alpha S_{z,z}=0. \tag{2.5b}$$

In (2.5a), H depends *a priori* on both F and A, but in virtue of the conditions $H=H(w)$ and $A=A(z)$, H, (and clearly F), must be independent of A. Consequently, it follows from the arbitrariness of A that its coefficient $(w\alpha_z+\beta_z)$ in (2.5a) must vanish identically. Therefore, $\alpha=k_3$ and $\beta=k_4$, for some constants k_3 and k_4. Substituting these values for α and β into (2.5a) and solving for H gives

$$H=F(k_3+k_4w)S_z^2/k_3, \tag{2.6}$$

and the condition $H_z=0$ forces S_z to be a constant function. Consequently, we must have $S=k_1z+k_2$, where k_1 and k_2 are arbitrary constants. A substitution of the expressions thus obtained for α,β and S into (2.5) completely determines B and H. The group G of equivalence transformations of (1.4) is therefore given by the linear transformations

$$x=k_1z+k_2,\qquad y=k_3w+k_4,\quad (k_1k_3=\delta\neq 0), \tag{2.7}$$

where the k_j, for $j=1,\dots,4$ are constants. On the other hand, the resulting induced transformations of the arbitrary functions A and F are given by

$$B=k_1A\circ L^{k_1,k_2},\qquad H=\frac{k_1^2}{k_3}F\circ L^{k_3,k_4} \tag{2.8}$$

where for every pair of constants p and q, $L^{p,q}$ is the linear function $L^{p,q}(a)=pa+q$. Consequently, the explicit form of the transformed equation (2.2) under (2.7) is given by

$$w''=k_1A(k_1z+k_2)+\frac{k_1^2}{k_3}F(k_3w+k_4), \tag{2.9}$$

and this shows in particular that for arbitrary values of the functions A and F, (1.4) has no nontrivial symmetries. Indeed, the symmetry group of the equation is a subgroup of its equivalence group, and (2.9) shows that (2.7) is a symmetry of (1.4) for every functions A and F only if

$$k_1z+k_2=z,\qquad \text{and}\qquad k_3w+k_4=w,$$

that is, only if (2.7) is the identity transformation.

An element of the family of equations of the form (1.4) may be labeled by the corresponding pair $\{A,F\}$ of coefficient functions, and by the above results two equations represented by the pairs $\{A,F\}$ and $\{B,H\}$ are equivalent under (2.1) if the coefficient functions are related by (2.8). By changing only the dependent variable in (1.4), we may

keep A fixed and transform only F, which induces among the coefficient functions F another equivalence relation that we denote by $\sim$. We have the following result about the latter equivalence relation.

Lemma 1. *Let a,b,c,r,s and n be given constants with $a,r \neq 0$ and $n \neq 1$. There are constants $\mu \neq 0$, and λ, θ such that the following hold in each case.*

(a) $r(ay+b)^n + cy + s \sim y^n + \lambda y + \theta$, *and* $ay^2 + by + c \sim y^2 + \theta$.
(b) *Let* $F = r e^{ay} + by + c$. *Then* $F \sim \mu e^y + \lambda y$ *for* $\lambda \neq 0$. *Else* $F \sim \mu e^y + \theta$.
(c) $a\ln(y) + by + c \sim \mu \ln(y) + \lambda y$.
(d) $ay\ln(y) + by + c \sim \mu y \ln(y) + \theta$.
(e) *Let* $F = cy + b$. *Then* $F \sim \mu y$ *for* $c \neq 0$, *else* $F \sim \theta$, *where* $\theta = 0$ *or* $\theta = 1$.

Proof. According to (2.8), we only need to show that in each case we can find constants k_3 and k_4 such that the given function F is equivalent to the indicated function H for some constants μ, λ and θ to be specified. This is achieved by finding for the given function, $F(y)$ say, the transformed function $H = F(k_3 y + k_4)/k_3$ which has the required form. In case (a) for example, letting

$$F = r(ay+b)^n + cy + s,$$

it is readily seen that by choosing $k_3 = (r a^n)^{1/(1-n)}$ and $k_4 = -b/a$, F is transformed into $H = y^n + \lambda y + \theta$, with $\lambda = c$ and $\theta = -(bc)/(ak_3) + s/k_3$. For the second part of (a) with $F = ay^2 + by + c$, the result follows by setting $k_3 = 1/a$, $k_4 = -b/(2a)$, which gives $\theta = (4ac - b^2)/4$. The other cases are treated in a similar manner. $\square$

3. Group classification

Equivalence transformations are often very helpful for the group classification of differential equations, because equivalent equations also have equivalent symmetry algebras, in the sense that one can be mapped onto the other by an invertible change of variables. However, in the actual case of Eq. (1.4), the transformations obtained in (2.7) and (2.8) are relatively weak, in the sense that they act on arbitrary functions only by mere scalings and translations and give rise in particular to an infinity of non equivalent equations. Nevertheless, we shall still be able to use them as a simplifying tool in our classification procedure of (1.4), which is based on a direct analysis of the determining equation of the symmetry algebra. In the sequel, the symbols M and m, as well as $k_1, k_2, \ldots$ will represent arbitrary constants. For a given function $Q = Q(a)$ with argument a, we shall write Q' for dQ/da. If we let

$$V = \xi(x,y)\,\partial_x + \phi(x,y)\,\partial_y \tag{3.1}$$

denote the generic generator of the symmetry algebra L of (1.4), then it readily follows from well-known procedures [16, 17] that the determining equations of L are given for arbitrary functions A and F by

$$\xi_{y,y} = 0 \tag{3.2a}$$
$$-\xi A' - A\xi_x - 3F\xi_y - \xi_{x,x} + 2\phi_{x,y} = 0 \tag{3.2b}$$
$$-\phi F' - 2F\xi_x - A\phi_x + F\phi_y + \phi_{x,x} = 0 \tag{3.2c}$$
$$-2A\xi_y - 2\xi_{x,y} + \phi_{y,y} = 0. \tag{3.2d}$$

From (3.2a) and (3.2d), it follows successively that

$$\xi = \alpha(x)y + \beta(x), \qquad \text{and} \qquad \phi = y^2\left(A\alpha(x) + \alpha'(x)\right) + y\,\sigma(x) + \tau(x), \tag{3.3}$$

for some functions α, β, σ and τ. Next, a substitution of (3.3) into (3.2) transforms (3.2b) and (3.2c) into

$$-3F\alpha + 3y\left(\alpha A' + A\alpha' + \alpha''\right) - \beta A' - A\beta' + 2\sigma' - \beta'' = 0 \tag{3.4a}$$

$$\begin{aligned} &-F'\left(-y\sigma - \tau + y^2\left(-A\alpha - \alpha'\right)\right) + F\left(2Ay\alpha + \sigma - 2\beta'\right) - A\tau' + \tau'' \\ &+ y\left(-A\sigma' + \sigma''\right) + y^2\left(-A\alpha A' - A^2\alpha' + 2A'\alpha' + \alpha A'' + \alpha'''\right) = 0 \end{aligned} \tag{3.4b}$$

Differentiating twice (3.4a) with respect to y shows that $-3F''\alpha = 0$, and thus we shall consider separately the cases $F'' = 0$ and $F'' \neq 0$.

3.1. **Case 1:** $F'' \neq 0$**.** We must have in this case $\alpha = 0$, and this reduces (3.4) to

$$-\beta A' - A\beta' + 2\sigma' - \beta'' = 0 \tag{3.5a}$$

$$(-y\sigma - \tau)F' + F\left(\sigma - 2\beta'\right) + y\left(-A\sigma' + \sigma''\right) - A\tau' + \tau'' = 0 \tag{3.5b}$$

Differentiating (3.5b) with respect to y twice yields

$$(\sigma + 2\beta')F'' + (y\sigma + \tau)F''' = 0. \tag{3.6}$$

If $F''' = 0$, then by the lemma we may assume that $F = y^2 + \theta$ where θ is a constant, and a substitution of the latter expression for F into (3.5b) leads after an expansion into powers of y to the equality

$$\sigma = -2\beta', \tag{3.7}$$

representing the vanishing of the coefficient of y^2. Substituting also the latter expression for σ into (3.5) and expanding again into powers of y gives

$$-\beta A' - A\beta' - 5\beta'' = 0 \tag{3.8a}$$

$$-4\theta\beta' - A\tau' + \tau'' = 0 \tag{3.8b}$$

$$\tau - A\beta'' + \beta''' = 0, \tag{3.8c}$$

Remark .

(a) It appears from the total orders of derivatives of unknown functions appearing in (3.8) that the maximal possible number κ_M of free parameters in the general solution cannot exceed five and therefore five is an upper bound for the dimension of the corresponding symmetry algebra L. However, the constraints in the system will in general reduce the size of κ_M.

(b) The order in which individual equations are solved and the corresponding solutions are substituted into the system doesn't matter, for they all yield the same solution. For (3.8) and all subsequent similar systems of equations, we shall therefore choose the order of integration that appears to be the most suitable for the solution.

For (3.8), a suitable order of integration consists in solving (3.8a) for β, substituting the result into (3.8c) to find τ, and using (3.8b) for the resulting compatibility condition on A.

This, together with (3.7), yields

$$\beta = e^{-\int \frac{A}{5}\,dx}\left(k_2 + k_1 \int e^{\int \frac{A}{5}\,dx}\,dx\right) \tag{3.9a}$$

$$\sigma = -2k_1 + \frac{2}{5}e^{-\int \frac{A}{5}\,dx}A\left(k_2 + k_1 \int e^{\int \frac{A}{5}\,dx}\,dx\right) \tag{3.9b}$$

$$\tau = \frac{1}{125}\left[-30k_1A^2 + 50k_1A' + e^{-\int \frac{A}{5}\,dx}\left(k_2 + k_1\int e^{\int \frac{A}{5}\,dx}\,dx\right)\left(6A^3 - 40AA' + 25A''\right)\right], \tag{3.9c}$$

and the compatibility condition on the coefficient A for the existence of any symmetry is given by

$$k_1(-20F_2E_2 + F_1E_1) + k_2E_1 = 0 \tag{3.10a}$$

where

$$\begin{aligned} E_1 = {} & 36A^5 - 900A^3A' + 2000A^2A'' \\ & + 625A\left(4\left(A'^2 + \theta\right) - 3A^{(3)}\right) + 625\left(-5A'A'' + A^{(4)}\right) \end{aligned} \tag{3.10b}$$

$$E_2 = 9A^4 - 180A^2A' + 275AA'' + 25\left(7A'^2 + 25\theta - 5A^{(3)}\right) \tag{3.10c}$$

$$F_1 = \int e^{\frac{1}{5}\int A\,dx}\,dx, \qquad \text{and} \qquad F_2 = e^{-\int A/5\,dx}. \tag{3.10d}$$

It is worthwhile recalling that the expression of the symmetry generator V of (3.1) is reduced in this case to

$$V = \beta(x)\,\partial_x + (y\,\sigma(x) + \tau(x))\,\partial_y.$$

For an explicit determination of the dimension of L, we note that since $E_1 = -5E_2' + 4AE_2$, where $E_2' = dE_2/dx$, it follows that $E_1 = 0$ if $E_2 = 0$, and hence L has dimension two if and only if $E_2 = 0$. On the other hand, L has dimension one if and only if exactly one of the following conditions hold

$$E_1 = 0, \qquad \text{or } -20F_2E_2 + F_1E_1 = 0, \tag{3.11}$$

the latter condition being an integro-differential equation. One-parameters families of solutions of (3.10a) indexed the arbitrary constant m are given by

$$A = p/(x+m), \qquad \text{for } \theta = 0, \text{ and } p = 0, -15, -10/3, -5/3 \tag{3.12a}$$

$$A = 5p\tan(px+m), \qquad \text{for } p = (-\sqrt{\theta}\,i/3)^{1/2}, \text{ and } k_1 = 0, \tag{3.12b}$$

and in case (3.12a) L has dimension two, while it has dimension one in case (3.12b), with generator

$$\cos\left[m + \frac{x\sqrt{-i\sqrt{\theta}}}{\sqrt{3}}\right]\partial_x + \frac{2\left(y + i\sqrt{\theta}\right)\sqrt{-i\sqrt{\theta}}\sin\left[m + \frac{x\sqrt{-i\sqrt{\theta}}}{\sqrt{3}}\right]}{\sqrt{3}}\partial_y. \tag{3.13}$$

We also have $\dim L = 1$ for $A = M$, but we can hardly describe all possibilities when L has exactly dimension one or two according to the values of A, because general solutions of $E_2 = 0$, where E_2 is given by (3.10c), or of (3.11) aren't available. This completes the classification problem when $F''' = 0$.

If $F''' \neq 0$, it follows from (3.6) that $(F''/F''')'' = 0$, and hence

$$F'''/F'' = 1/(a_1y + a_2)$$

for some constants a_1 and a_2, and we have to consider separately the two possibilities $a_1 \neq 0$, and $a_1 = 0$ (but $a_2 \neq 0$). All these possibilities lead to the following possible canonical forms for F :

$$\text{Case (i):} \quad F = \mu e^y + \lambda y \quad (\lambda \neq 0), \qquad \text{Case (ii):} \quad F = \mu e^y + \theta \tag{3.14a}$$

$$\text{Case (iii):} \quad F = \mu \ln(y) + \lambda y, \qquad \text{Case (iv):} \quad F = \mu y \ln(y) + \theta \tag{3.14b}$$

$$\text{Case (v):} \quad F = y^n + \lambda y + \theta, \ (n \neq 0, 1, 2), \tag{3.14c}$$

where λ, θ and μ are constants with $\mu \neq 0$, and where the first two cases (i) and (ii) correspond to $a_1 = 0$, while the remaining cases correspond to $a_1 \neq 0$.

In the cases (i), (iii), and the cases (iv) and (v) with $\theta \neq 0$, it is readily found that the only symmetry is $V = \partial_x$, provided that the coefficient A is a constant function.

In case (ii), when $\theta = 0$ we have $F = \mu e^y$ and a substitution of this expression into (3.5) shows that L is generated by

$$V = \left(k_2 + k_3 x + k_1 \int\int e^{\int A dx} dx dx\right) \partial_x + \left(-2k_3 - 2k_1 \int e^{\int A dx} dx\right) \partial_y \tag{3.15}$$

provided that the compatibility condition

$$-k_2 A' - k_3 \left(A + xA'\right) + k_1 \left(-e^{\int A\, dx} - A \int e^{\int A\, dx} dx - A' \int\int e^{\int A\, dx} dx\, dx\right) = 0 \tag{3.16}$$

on the coefficient A is satisfied. An analysis of (3.16) shows that $\dim L = 2$ for $A = 0$ or $A = -1/x$, and $\dim L = 1$ for $A = M$, $M \neq 0$ or $A = M/x$, $M \neq -1$. Else L is zero-dimensional.

When $F = \mu e^y + \theta$, and $\theta \neq 0$, a substitution of this expression into (3.5) shows that we must have $\alpha = \sigma = 0$, and $\tau = -2\beta'$. All these expressions for F, α, σ and τ reduce (3.5) to

$$-\beta A' - A\beta' - \beta'' = 0 \tag{3.17a}$$

$$\theta \beta' - A\beta'' + \beta''' = 0, \tag{3.17b}$$

and we find the solution by solving (3.17a) for β and substituting the result into the second equation. In this way we find the corresponding symmetry generator to be of the form

$$V = F_2(k_2 + k_1 F_1)\, \partial_x + (-2k_1 + 2F_2 A(k_2 + k_1 F_1))\, \partial_y \tag{3.18a}$$

where

$$F_1 = \int e^{\int A\, dx} dx, \qquad F_2 = e^{-\int A\, dx} \tag{3.18b}$$

and the compatibility condition on A is given by

$$k_1 \left(-E_4 + F_2 F_1 E_3\right) + k_2 F_2 E_3 = 0 \tag{3.19a}$$

where

$$E_3 = 2A^3 + A\left(\theta - 4A'\right) + A'', \quad \text{and} \quad E_4 = \theta + 2A^2 - 2A'. \tag{3.19b}$$

We have $E_4' + 2E_3 = 2AE_4$ where $E_4' = dE_4/dx$, and thus we readily see that L has dimension two if and only if $E_4 = 0$, that is, if and only if

$$A = \sqrt{\frac{\theta}{2}} \tan\left(\sqrt{\frac{\theta}{2}}(x + 2m)\right),$$

where m is a constant. On the other hand, L has dimension one if A satisfies exactly one of the conditions

$$E_3 = 0, \quad \text{or } -E_4 + F_1F_2E_3 = 0. \tag{3.20}$$

More explicitly, the latter equality is an integro-differential equation of the form

$$-2\left(\theta + 2A^2 - 2A_x\right) + 2e^{-\int A\,dx}\left(\int e^{\int A\,dx}dx\right)\left(2A^3 + A\left(\theta - 4A_x\right) + A_{x,x}\right) = 0. \tag{3.21}$$

In case (iv) with $\theta = 0$, we have $F = \mu y \ln(y)$, and a substitution of this expression for F into (3.5) shows that $\alpha = \tau = 0$, and $\beta = k_1$, and the remaining conditions on A and σ are given by

$$-k_1 A' + 2\sigma' = 0 \tag{3.22a}$$

$$-\mu\sigma - A\sigma' + \sigma'' = 0, \tag{3.22b}$$

Since for every function A (3.22) always consists of a first-order and a second-order ODE, its solution will depend on at most one arbitrary constant, and L will therefore have at most dimension two in this case, with corresponding symmetry generator

$$V = k_1\,\partial_x + y\sigma\,\partial_y.$$

If we solve (3.22a) for either A or σ and substitute the result in (3.22b), then neither the resulting equation, nor (3.22b) itself is tractable. It is therefore not possible to describe all the possible dimensions of L according to the values of A. It is however clear that for $A = M$, we have $\sigma = 0$, and $V = \partial_x$. More generally, the compatibility condition on A is given by

$$2k_2\mu + k_1 A(\mu + A') - k_1 A'' = 0 \tag{3.23}$$

where k_2 is another constant.

For case (v), when $\theta = \lambda = 0$, we have $F = y^n$ $(n \neq 0, 1, 2)$, and the corresponding generator of L is given by

$$V = \left(k_2 + k_3 x + k_1 \int\int e^{\int A dx} dx dx\right)\partial_x - \left(\frac{2k_3 y}{n-1} + \frac{2k_1 y \int e^{\int A dx} dx}{n-1}\right)\partial_y$$

while the related compatibility condition on A is given by

$$\begin{aligned} &-k_2(n-1)A' - k_3(n-1)\left(A + xA'\right) \\ &-k_1\left(e^{\int A\,dx}(3+n) + (n-1)A\int e^{\int A\,dx}dx + (n-1)A'\int\int e^{\int A\,dx}dx\,dx\right) = 0. \end{aligned} \tag{3.24}$$

It thus follows that L has dimension three if and only if $A = 0$ and $n = -3$. On the other hand, we have $\dim L = 2$ if

$$\begin{aligned} &A = -\frac{(n+3)/x}{n+1}, \quad n \neq -1, -3, && \text{or if} \\ &A = M, \quad \text{with} \quad M \neq 0 \quad \text{and} \quad n = -1, && \text{or if} \\ &A = 0 \quad \text{and} \quad n \neq -3. \end{aligned}$$

We also have $\dim L = 1$ if

$$A = \begin{cases} M/x, & \text{with } M \neq 0, -(3+n)/(n+1) \\ M, & \text{with } M \neq 0, \quad \text{and} \quad n \neq -1 \end{cases}$$

or if

$$A \neq -\frac{(n+3)/x}{n+1}, \quad \text{and} \tag{3.25a}$$

$$0 = e^{\int A\,dx}(3+n) + (n-1)A\int e^{\int A\,dx}\,dx + (n-1)A'\int\int e^{\int A\,dx}\,dx\,dx \tag{3.25b}$$

In case (v) with $\lambda \neq 0$, the symmetry generator is given by

$$V = \beta\,\partial_x - \frac{2y\beta'}{n-1}\,\partial_y, \tag{3.26}$$

where β is determined together with the coefficient A by the equation

$$\beta A' + A\beta' - (3+n)\beta'' = 0 \tag{3.27a}$$

$$(n-1)\lambda\beta' - A\beta'' + \beta''' = 0. \tag{3.27b}$$

It follows from (3.27a) that for $n \neq -3$ we have

$$\beta = (k_2 + k_1 F_1)F_2 \tag{3.28a}$$

where

$$F_1 = \int \exp\left(\frac{(n-1)\int A\,dx}{3+n}\right)dx \quad \text{and} \quad F_2 = -\exp\left(\frac{(n-1)\int A\,dx}{3+n}\right) \tag{3.28b}$$

and the corresponding compatibility conditions for A are given by

$$k_2F_2E_5 + k_1\left((n+3)E_6 + F_1F_2E_5\right) = 0, \tag{3.29a}$$

where

$$E_5 = 2A^3\left(-1+n^2\right) + A(3+n)\left((-1+n)(3+n)\lambda - 4nA'\right) + (3+n)^2A'' \tag{3.29b}$$

$$E_6 = -2A^2(1+n) + (3+n)\left((-3-n)\lambda + 2A'\right). \tag{3.29c}$$

In this case we have

$$2E_5 - (3+n)E_6' + 2(n-1)AE_6 = 0,$$

where $E_6' = dE_6/dx$. Consequently, we have $\dim L = 2$ if and only if

$$A = \begin{cases} \frac{(3+n)\sqrt{\lambda}}{\sqrt{2(1+n)}}\tan\left[\frac{\sqrt{\lambda(n+1)}(x+2(3+n)m)}{\sqrt{2}}\right], & \text{for } n \neq -1 \\ \lambda x + m, & \text{for } n = -1 \end{cases}$$

where m is a constant parameter. Clearly, we have $\dim L = 1$ if exactly one of the following conditions hold:

$$E_5 = 0, \quad \text{or} \quad (n+3)E_6 + F_1F_2E_5 = 0. \tag{3.30}$$

However, we cannot obtain in general all explicit expressions of A for which $\dim L = 1$, although here again for $A = M$ we have $\dim L = 1$, and $V = \partial_x$.

For $n = -3$, Eq. (3.27) reduces to

$$\beta A' + A\beta' = 0 \tag{3.31a}$$

$$-4\lambda\beta' - A\beta'' + \beta''' = 0. \tag{3.31b}$$

For $A = 0$, (3.31a) vanishes identically, and (3.31) reduces to (3.31b) in which $A = 0$. Solving the resulting equation for β and substituting the result into (3.26) yields the generator

$$V = \left[k_3 + \frac{e^{-2x\sqrt{\lambda}}\left(k_1 e^{4x\sqrt{\lambda}} - k_2\right)}{2\sqrt{\lambda}} \right] \partial_x + \frac{1}{2} e^{-2x\sqrt{\lambda}} \left(k_1 e^{4x\sqrt{\lambda}} + k_2\right) y\,\partial_y \tag{3.32}$$

where k_1, k_2 and k_3 are arbitrary constants and this shows in particular that L has dimension three in this case. For $A \neq 0$, the general solution of (3.31) will depend on at most one arbitrary parameter, and hence L will have dimension at most one. If we set for instance $A = M$ in (3.31), where M is a nonzero constant, this gives $\beta = k_1$, where k_1 is another constant, with corresponding symmetry generator $V = k_1\,\partial_x$. However, we cannot describe all other solutions to (3.31), and hence we cannot describe in this case all values of A for which the dimension of L takes on the value one. Indeed, from (3.31a) we have $A = k_1/\beta$, and substituting this into (3.31b) gives

$$-4\lambda\beta' - k_1\beta''/\beta + \beta^{(3)} = 0,$$

which is an equation for which the general solution is not available. On the other hand we can look for one-dimensional subalgebras of L by solving (3.31a) for β. This gives $\beta = k_1/A$, and the corresponding compatibility condition for A takes the form

$$2A'^2 + \frac{6A'^3}{A^2} - AA'' + A'\left(-4\lambda - \frac{6A''}{A}\right) + A''' = 0. \tag{3.33}$$

However, for this nonlinear equation we can only obtain the particular solution $A = const$.

3.2. **Case 2: $F'' = 0$.** By a linearization test for second-order ODEs due to Lie [18], (1.4) is linearizable for a given function F if and only if F is linear, and we are therefore in the linear case of (1.4). It then follows from an already cited result of Lie [1] that (1.4) has a symmetry algebra of maximal dimension eight.

We now undertake to verify that for linear $F = \lambda y + \theta$, the dimension of L is always eight, regardless of the value of A. It follows from the equivalence relations on functions F that we may assume that $F = \lambda y$, $\lambda \neq 0$ or $F = \theta$. For $F = \lambda y$, a substitution of this expression into (3.4) shows that

$$\sigma = k_1 + \int \frac{1}{2}(A'\beta + A\beta' + \beta'')dx. \tag{3.34}$$

When the latter expression for σ is also substituted into (3.4), we obtain after a split into powers of y the following equations

$$\alpha\left(-\lambda + A'\right) + A\alpha' + \alpha'' = 0 \tag{3.35a}$$

$$-\lambda\tau - A\tau' + \tau'' = 0 \tag{3.35b}$$

$$A\alpha\lambda - A\alpha A' - A^2\alpha' + \left(\left(-\lambda + A'\right)\alpha' + \alpha A'' + \alpha'''\right) = 0 \tag{3.35c}$$

$$-A\beta A' - A^2\beta' + 2\left(-2\lambda + A'\right)\beta' + \beta A'' + \beta''' = 0 \tag{3.35d}$$

We note that only (3.35a) and (3.35c) are dependent equations, and we denote by E_7 and E_8 the left hand side of (3.35c) and (3.35a), respectively. We have $E_7 = E_8' - AE_8$, where $E_8' = dE_8/dx$, and this shows that (3.35a) alone is equivalent to the system consisting of the two equations (3.35a) and (3.35c). Therefore, to find α, τ and β, we only need to solve the independent linear equations (3.35a), (3.35b), and (3.35d). The sum of orders of these independent equations together with the arbitrary constant k_1 in (3.34) is exactly eight and therefore yields an 8-parameter symmetry algebra, regardless of the value of A. However,

TABLE 1. Classification results for the equation $y'' = A(x)y' + F(y)$.

F	A	$\dim L$	Generator V
μe^y	0	2	$(k_1 + k_2 x)\partial_x - 2k_2\partial_y$
	$-1/x$	2	
	$M/x, \quad M \neq -1$	1	$x\partial_x - 2\partial_y$
$\mu e^y + \theta,\ \theta \neq 0$	$\sqrt{\frac{\theta}{2}}\tan\left(\sqrt{\frac{\theta}{2}}(x+2m)\right)$	2	
	As given by (3.20)	1	
$\mu y \ln(y)$	As given by (3.23)	1 or 2	
y^2	$p/(x+m), \quad p = 0, -15, \frac{-10}{3}, \frac{-5}{3}$	2	
	As given by $E_2 = 0$	2	
	As given by (3.11), with $\theta = 0$	1	
$y^2 + \theta,\ \theta \neq 0$	$5(\sqrt{\theta}i/3)^{1/2}\tan((\sqrt{\theta}i/3)^{1/2}(x+m))$	1	
	As given by $E_2 = 0$	2	
	As given by (3.11)	1	
y^{-1}	$M,\ M \neq 0$	2	
	$M/x, \quad M \neq 0$	1	$x\partial_x + y\partial_y$
y^{-3}	0	3	
	$M/x, \quad M \neq 0$	1	$2x\partial_x + y\partial_y$
	As given by (3.25), with $n = -3$	1	
y^n, $(n \neq -3)$	$\frac{-(n+3)/x}{(n+1)},\ n \neq -1$	2	$(k_2 + k_1 x)\partial_x - \frac{2k_1}{n-1}\partial_y$
	0	2	
	$M/x, \quad M \neq 0, -\frac{n+3}{n+1},\ n \neq -1$	1	$(n-1)x\partial_x - 2y\partial_y$
	As given by (3.25)	1	
$y^{-1} + \lambda y,\ \lambda \neq 0$	$\lambda x + m$	2	
	As given by (3.30), with $n = -1$	1	
$y^{-3} + \lambda y,\ \lambda \neq 0$	0	3	
	As given by (3.33)	1	
$y^n + \lambda y,\ \lambda \neq 0$, $(n \neq -3, -1)$	$\frac{(3+n)\sqrt{\lambda}}{\sqrt{2(1+n)}}\tan\left[\frac{\sqrt{\lambda(n+1)}(x+2(3+n)m)}{\sqrt{2}}\right]$	2	
	As given by (3.30)	1	
$F(y)$	M	1	∂_x

we cannot find explicit expressions for the generator V for arbitrary values of A and λ, because this involves solving equations of the form (3.35b) for which the general solution is not available. A similar analysis holds when $F = \theta$ is a constant function.

3.3. **Summary.** We have represented the classification results for the nonlinear case of (1.4) in Table 1, in which the first column indicates admissible canonical forms of F for which symmetries exist, while the second column indicates the corresponding expression of the function A. When A is given by complicated nonlinear or linear equations for which the solution is not available, the required expression is replaced in the column the determining equation for A given in the text. The third column gives the generator V of L, but only for those cases for which the explicit expression for V is available, and when its size is sufficiently small to fit in the table. However, explicit forms of the symmetry generator V from which symmetries can be readily calculated for given values of A is provided every time in the text for every possible canonical form F, and most of the generators V with a relatively prominent size have been determined in the text whenever A was known.

In the last row, the pair $\{M, F(y)\}$ represents an equation with an arbitrary admissible function $F(y)$ and with $A = M$, provided that such a function is not represented else where in the table. For instance for $F(y) = y^{-1}$ and $A = M$, $M \neq 0$, the equation is represented in the row with $F = y^{-1}$, while for $A = 0$, the corresponding equation is represented in the block of rows with $F = y^n$, $(n \neq -3)$. Indeed, table rows represent non equivalent equations.

4. CONCLUDING REMARKS

In this paper, we have given a group classification of equations of the form (1.4) and shown that the only admissible canonical forms of F admitting symmetries are given by functions listed in (1.5). Moreover, we have shown that in the nonlinear case, the maximal dimension of the symmetry algebra L achieved is three, which coincide with the result obtained in [2] for equations of the form (1.2). We have also been able to determine for any canonical form F explicit expressions of A for which L has a given dimension n, where $0 \leq n \leq 3$, with a few exceptions which apply mostly for cases of one dimensional subalgebras, and rarely for two-dimensional subalgebras. Indeed, we have not been able to provide explicit expressions of A for which $\dim L = 2$ only for $F = \mu e^y + \theta$, and for $F = y^2 + \theta$ although in the latter case we have given some one-parameter families of functions A for which the equality $\dim L = 2$ occurs. These difficulties are due to the fact that general solutions of the related determining equations for A, such as $E_2 = 0$, where E_2 is given by (3.10c), is not available.

For cases one one-dimensional symmetry algebras L, the difficulty with finding an explicit expression for the corresponding function A is often due to the fact that determining equations are quite often complicated integro-differential equations or nonlinear equations of the form (3.33). On the other hand, this difficulty as well as the impossibility to give an explicit expression of the symmetry generator in the linear case for arbitrary values of A is due to the fact that the general solution is not known for linear equations similar to (3.35b) and of the form

$$\beta'' + f(x)\beta' + g(x)\beta = 0,$$

in which f is an arbitrary function and g is either another arbitrary function or a nonzero constant. We note however that equations of the latter form can always be reduced to the most common canonical form

$$w'' + h(x)w = 0, \qquad \text{with } h = -(f^2 - 4g + 2f')/4,$$

by a change of the dependent variable of the form $\beta = we^{-(1/2)\int f dx}$. Although the reduced equation depends on fewer arbitrary functions, the difficulty with solving it remains essentially the same for arbitrary functions h. However, using the provided determining equations, this classification can nevertheless readily provide the explicit symmetry generator V for any given pair $\{A, F\}$ of labeling functions, or tell when a symmetry does not exist.

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